A FUSION ALGORITHM FOR SOLVING THE HIDDEN SHIFT PROBLEM IN FINITE ABELIAN GROUPS

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THE HIDDEN SHIFT PROBLEM

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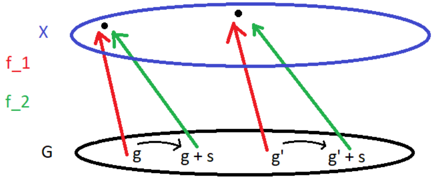
Definition

Let $(G, +)$ be an abelian group, X be a set and $f_1, f_2 : G \to X$ be injective functions for which there is a $s \in G$ such that $f_1(g) = f_2(g + s)$ for all $g \in G$. Find s.



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- System of equations $\to$ recover s

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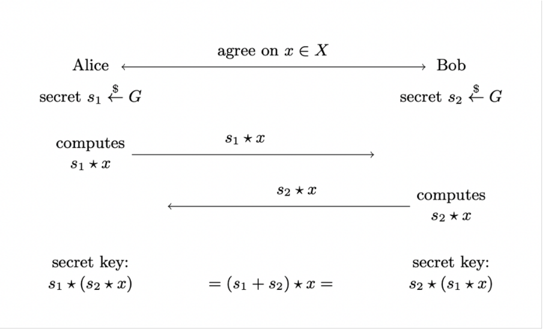
Abelian group G acts on set X



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STANDARD METHOD

- Apply f_1, f_2 to a superposition of all group elements
- measure to obtain

$$\frac{1}{\sqrt{2}}(|g\rangle + |g+s\rangle)$$

- Perform Abelian Quantum Fourier transform to obtain

$$\frac{1}{\sqrt{2}}(|0\rangle + \chi(s)|1\rangle)$$

- $\chi : G \longrightarrow \mathbb{C}^\times$ is a character

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  - Kuperberg: collimation sieve using QRACM
  - Peikert: adaptation of collimation sieve on most significant bit

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- Goal: combine characters to get $(a_1, a_2, 0, 0)$

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- After enough equations, we can recover s_1, s_2

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- Use method Simon's problem to recover s_3, s_4

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- thin out to obtain regular phase vector supported on a subgroup H of $G^\vee$.
- Apply Fourier transform to retrieve s mod $\ker H$.
- Reduces hidden shift problem to $\ker H$.

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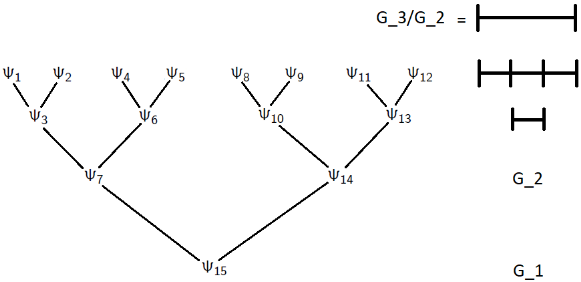

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- Apply **Collimation Algorithm** to retrieve s entirely

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  - Can this be generalized to other torsion, while being kept memory-friendly?

THE END

Thank you for your attention