Memory Optimization Techniques for Computing Discrete Logarithms in Compressed SIKE

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Outline

1. Isogenies, SIKE, and compressed SIKE
2. Mathematical background and the problem definition
3. Contributions
4. Implementation results

Notation

- $p = {\ell_A}^{e_A}{\ell_B}^{e_B} - 1$ is prime, $\mathbb{F}_q$ is a finite field of size $q = p^2$
  - $\ell_A = 2$, $\ell_B = 3$
  - $(e_A, e_B) \in \{(216, 137), (250, 159), (305, 192), (372, 239)\}$
  - $\mathbb{F}_q = \mathbb{F}_p[i]/\langle i^2 + 1\rangle$
- E is a supersingular elliptic curve defined over $\mathbb{F}_q$
  - $E/\mathbb{F}_q: \ \{(x, y): \ y^2 = x^3 + 6x^2 + x\} \cup \mathcal{O}$
- $E(\mathbb{F}_q)$ denotes the set of $\mathbb{F}_q$-points on E
  - $|E(\mathbb{F}_q)| = (p + 1)^2$
- $E[\ell^e]$ denotes the ℓ^e-torsion group of E ($\ell^e P = \mathcal{O}$ for $P \in E[\ell^e]$)
  - $E[\ell^e] \cong \mathbb{Z}_{\ell^e} \oplus \mathbb{Z}_{\ell^e}$
  - $E[{\ell_A}^{e_A}] = \langle P_A, Q_A\rangle$ and $E[{\ell_B}^{e_B}] = \langle P_B, Q_B\rangle$

Isogenies

- Let E_1, E_2 be two elliptic curves over $\mathbb{F}_q$
- An *isogeny* $\phi : E_1 \to E_2$ is a non-constant rational map such that
  - ϕ is defined over $\mathbb{F}_q$
  - $\phi(\mathcal{O}) = \mathcal{O}$
- $\phi : E_1(\mathbb{F}_q) \to E_2(\mathbb{F}_q)$ is a group homomorphism
- $\ker(\phi) = \{P \in E_1 : \phi(P) = \mathcal{O}\}, \ E_2 \cong E_1/\ker(\phi)$
- Conversely, given a subgroup K of $E_1(\mathbb{F}_q)$, there exists (unique up to isomorphism)
$$\phi : E_1 \to E_2, \quad \text{where } \ker(\phi) = K$$
- Given K, ϕ and E_2 can be expressed explicitly (may not be efficient)

SIKE: Supersingular Isogeny Key Encapsulation

- SIKE is based on the supersingular isogeny Diffie-Hellman (SIDH) protocol (Jao and De Feo, 2011)
- SIKE was submitted to the NIST post-quantum cryptography standardization process (2017)
- SIKE was announced as one of the five alternate candidates in the public key encryption and key establishment category (2020)
- In SIKE, two parties A and B compute a shared secret key, which is essentially the j-invariant of the two isomorphic curves E_{AB} and E_{BA}: $j = j(E_{AB}) = j(E_{BA})$

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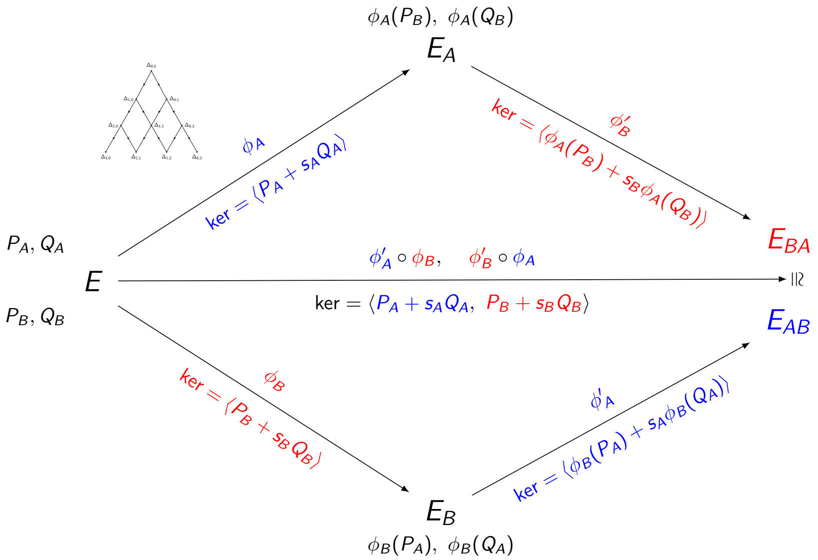
A basic setting for SIKE:

- **Public parameters:** $p = 2^{e_2}3^{e_3} - 1$, $E : y^2 = x^3 + 6x^2 + x$, $E[2^{e_2}] = \langle P_A, Q_A \rangle$, $E[3^{e_3}] = \langle P_B, Q_B \rangle$
- **Secret key of A:** $s_A \in \mathbb{Z}_{2^{e_2}}$
- **Public key of A:** E_A, $\phi_A(P_B)$ $\phi_A(Q_B)$, where $\phi_A : E \to E_A$ is a secret isogeny with $\ker(\phi_A) = \langle P_A + s_A Q_A \rangle$
- The secret/public keys of B are defined similarly

Computations in SIKE



Compressing SIKE public keys

- Naive method:
  - Public key of A: $E_A : y^2 = x^3 + ax + b$, $a, b \in \mathbb{F}_q$; $P = \phi_A(P_B) = (x_P, y_P)$, $Q = \phi_A(Q_B) = (x_Q, y_Q) \in E[\ell^e]$
  - $a, b, x_P, y_P \in \mathbb{F}_q$ can be represented by $8 \log p$ bits

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- Optimizations (Azarderakhsh *et al.*, 2016; Costello *et al.*, 2017; Gustavo *et al.*, 2018; Naehrig and Renes *et al.*, 2019; Pereira *et al.*, 2020):
  - Replace a, b by the j-invariant of E_A: $4 \log p \to 2 \log p$
  - Fix a basis for $E_A[\ell^e] = \langle R, S \rangle$ and rewrite

$$P = a_P R + b_P S; \;\; Q = a_Q R + b_Q S$$

  - Replace x_P, x_Q by $a_P, b_P, a_Q, b_Q \in \mathbb{Z}_{\ell^e}$: $4 \log p \to 2 \log p$ ($\ell^e \approx \sqrt{p}$)
  - Rescale $[a_P, b_P, a_Q, b_Q]$ by a_P^{-1}: $2 \log p \to 1.5 \log p$
  - Public key compression: $8 \log p \to 3.5 \log p$

Computational overheads in compressing public keys

- How to compute a_P, b_P, a_Q, b_Q?
- Compute g and h, where

$$g = e(R, S)$$
$$h = e(P, S) = e(a_P R + b_P S, S) = g^{a_P},$$

where $e : E_A[\ell^e] \times E_A[\ell^e] \to \mu_{\ell^e}$ is a bilinear pairing function, and μ_{ℓ^e} is the multiplicative group of order ℓ^e

- Compute a_P by solving the discrete logarithm of h base g
- Similarly, compute b_P, a_Q, b_Q
- Since ℓ is small, we use the Pohlig-Hellman algorithm to solve logarithms

The Discrete logarithm problem (DLP)

- p is prime, $p \equiv 3 \pmod 4$, $\mathbb{F}_q = \mathbb{F}_p[i]/\langle i^2 + 1 \rangle$
- $\mathbb{G}$ is the order-$(p+1)$ cyclotomic subgroup of $\mathbb{F}_q^*$
- For positive ℓ and ω with $\ell^\omega \mid (p+1)$, $\mathbb{G}_{\ell,\omega}$ is the order-ℓ^ω subgroup of $\mathbb{G}$
- $\mathbb{G}_{\ell,e} = \langle g \rangle$ is the largest of $\mathbb{G}_{\ell,\omega} = \langle \rho \rangle$, and $\rho = g^{\ell^{e-\omega}}$
- In particular, we are interested in:
  - $p = 2^{e_A} 3^{e_B} - 1$
  - $\ell = 2$ and $e \in \{216, 250, 305, 372\}$
  - $\ell = 3$ and $e \in \{137, 159, 192, 239\}$

Problem

Given $\mathbb{G}_{\ell,e} = \langle g \rangle$ and $h \in \mathbb{G}_{\ell,e}$, find $d \in \mathbb{Z}_{\ell^e}$ such that $g^d = h$

Remark

Inverses can be computed for free in $\mathbb{G}$:

$$h = (a + bi) \in \mathbb{G} \Rightarrow 1 = h^{p+1} = (a + bi)^p (a + pi) = (a - bi)(a + bi)$$

The Pohlig-Hellman (PH) algorithm

Problem

Given $\mathbb{G}_{\ell,e} = \langle g \rangle$ *and* $h \in \mathbb{G}_{\ell,e}$*, find* $d \in \mathbb{Z}_{\ell^e}$ *such that* $g^d = h$*.*

1. For $0 \leq k < e$ and $0 \leq d < \ell$, precompute and store the table

$$T[k][d] = g^{-d\ell^k}$$

2. Write $d = \sum_{i=0}^{e-1} d_i \ell^i$, $d_i \in [0, \ell)$ and use $g^{\ell^e} = 1$
3. Compute

$$\Delta_{0,0} = h = g^{\sum_{i=0}^{e-1} d_i \ell^i}, \; \Delta_{e-1,0} = \Delta_{0,0}^{\ell^{e-1}} = (g^{\ell^{e-1}})^{d_0} \in \mathbb{G}_{\ell,1}$$

4. Determine d_0: if $\Delta_{e-1,0} == T[e-1][d]^{-1}$ then $d_0 \leftarrow d$

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4. Determine d_0: if $\Delta_{e-1,0} == T[e-1][d]^{-1}$ then $d_0 \leftarrow d$
5. Compute

$$\Delta_{0,1} = \Delta_{0,0} T[0][d_0] = g^{\sum_{i=1}^{e-1} d_i \ell^i}, \Delta_{e-2,1} = \Delta_{0,1}^{\ell^{e-2}} = (g^{\ell^{e-1}})^{d_1}$$

6. Determine d_1: if $\Delta_{e-2,1} == T[e-1][d]^{-1}$ then $d_1 \leftarrow d$

The PH algorithm

- For $k = 1, ..., e-1$, compute

$$\Delta_{0,k} = \Delta_{0,k-1} T[k-1][d_{k-1}] = g^{\sum_{i=k}^{e-1} d_i \ell^i}, \Delta_{e-1-k,k} = \Delta_{0,k}^{\ell^{e-1-k}} = (g^{\ell^{e-1}})^{d_k}$$

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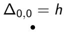

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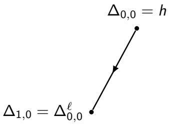

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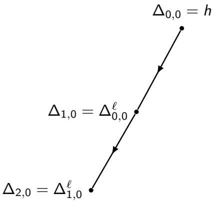

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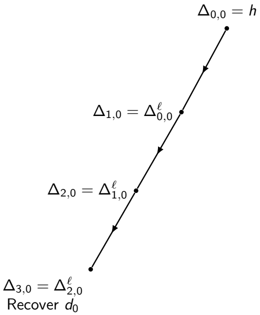

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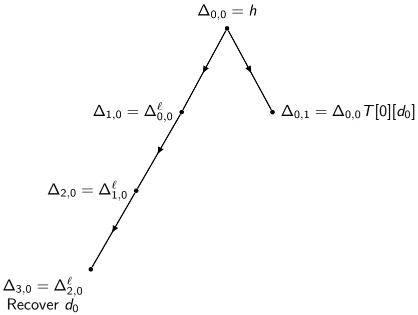

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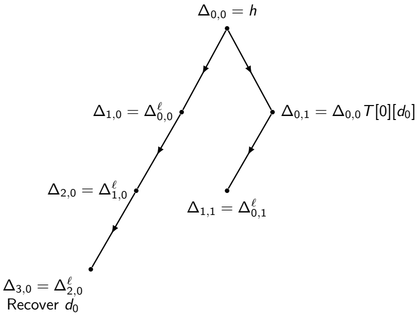

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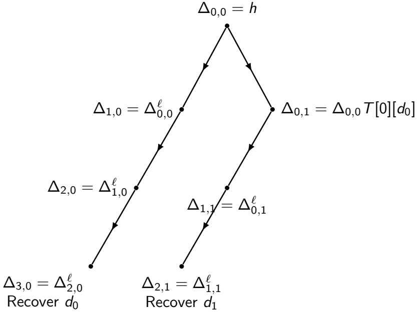

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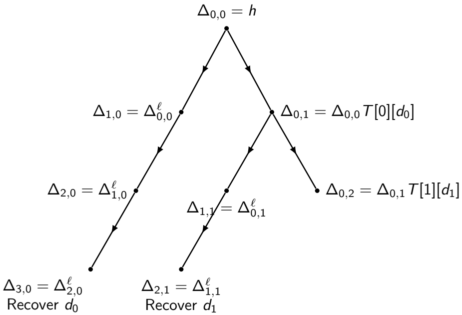

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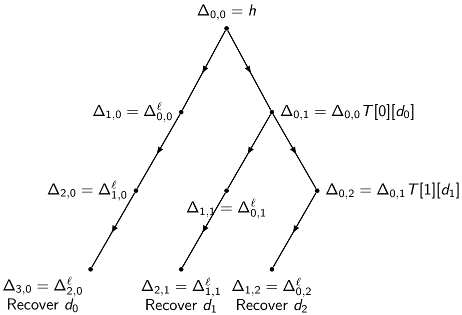

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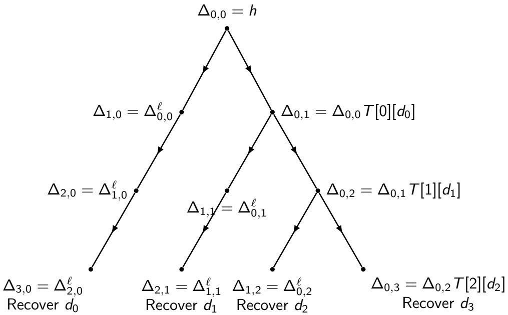
- Determine d_k: if $\Delta_{e-1-k,k} == T[e-1][d]^{-1}$ then $d_k \leftarrow d$
- **Cost:** $e(e-1)/2 = 6$ ℓ-exponentiations, $(e-1) = 3$ group multiplications, $e = 4$ table look ups
- **Storage:** $e\ell$ group elements in $\mathbb{F}_q^*$: $2e\ell \log p$ bits



An Optimal PH algorithm

- Optimal trade-off between the multiplication and ℓ-exponentiation (Gustavo *et al.*, 2018):
- Among all possible traversing strategies, find one with minimum cost

$$\Delta_{0,0} = h$$

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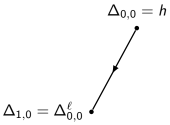

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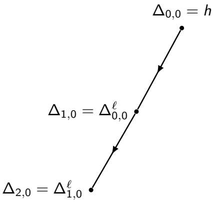

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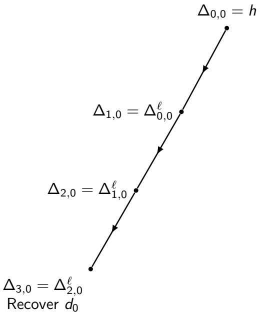

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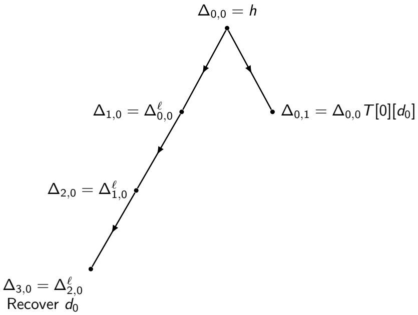

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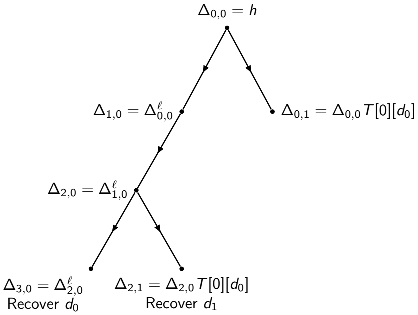

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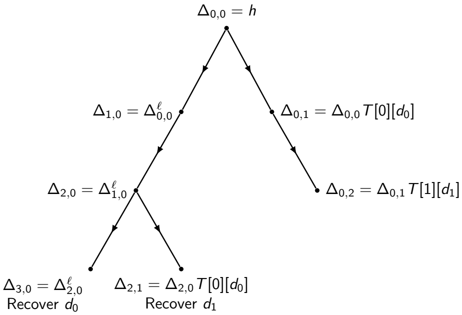

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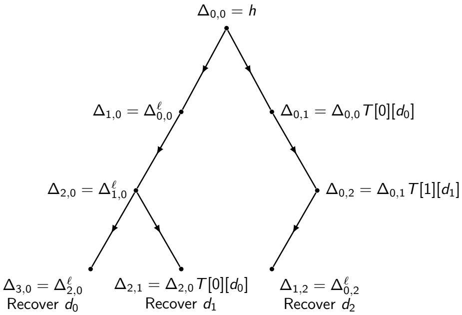

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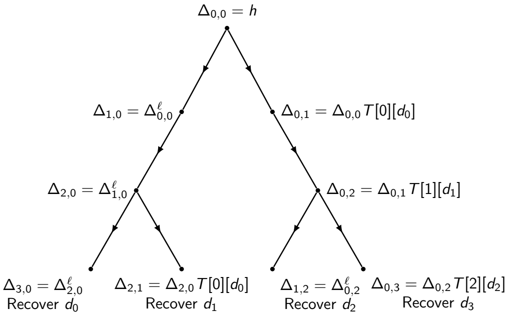
- **Storage:** $e\ell$ group elements in $\mathbb{F}_q^*$: $2e\ell \log p$ bits



Optimization 1: Signed-digits in the exponent

- $T[k][d] = g^{-d\ell^k},\ 0 \le k < e,\ 0 \le d < \ell$, stores $e\ell$ group elements
- Discard the entries with $d = 0$
- Write $d = \sum_{i=0}^{e-1} d_i' \ell^i$ for $d_i \in [-(\ell-1)/2, (\ell-1)/2]$ and modify T to T^{sgn} as

$$T^{\text{sgn}}[k][d] = g^{-d\ell^k},\ 0 \le k < e,\ \boxed{0 \le d \le (\ell-1)/2}$$

- Revise the step to determine d_k as
  1. If $\Delta_{e-1-k,k} == T^{\text{sgn}}[e-1][d]^{-1}$ then $d_k' \leftarrow d$
  2. If $\Delta_{e-1-k,k} == T^{\text{sgn}}[e-1][d]$ then $d_k' \leftarrow d - \ell$
- Updating $\Delta_{j,k+1} \leftarrow \Delta_{j,k} T[j+k][d_k]$ is revised similarly:
  1. If $d_k' > 0$, then $\Delta_{j,k+1} \leftarrow \Delta_{j,k} T^{\text{sgn}}[j+k][d_k']$
  2. If $d_k' < 0$, then $\Delta_{j,k+1} \leftarrow \Delta_{j,k} T^{\text{sgn}}[j+k][-d_k']^{-1}$

Remark

The table size is reduced by a factor of 2.

Optimization 2: Torus representations and arithmetic in $\mathbb{G}$

- $\mathbb{G} = \{a + bi : \ a, b \in \mathbb{F}_p, \ a^2 + b^2 = 1\}$ is the order-$(p+1)$ cyclotomic subgroup of ${\mathbb{F}_{p^2}}^*$

- Torus representation of $\mathbb{G}$ yields (Rubin and Silverberg, 2008):

$$\mathbb{G} = \{1\} \cup \left\{ \frac{\alpha + i}{\alpha - i} : \ \alpha \in \mathbb{F}_p \right\},$$

  where $a + bi = (\alpha + i)/(\alpha - i)$ with $\alpha = (a+1)/b$

- This compressed representation preserves group multiplication

- For $C : \mathbb{G} \setminus \{1, -1\} \to \mathbb{F}_p$ with $C(a + bi) = (a+1)/b$, we have

$$C(a+bi) = \alpha, C(c+di) = \beta, \alpha+\beta \neq 0 : \quad \boxed{C((a+bi)(c+di)) = \frac{\alpha\beta - 1}{\alpha + \beta}}$$

Torus representations and arithmetic

- Using projective coordinates $[x:y] := (x+yi)/(x-yi)$, we can write

$$\mathbb{G} = \{a+bi: \ a,b \in \mathbb{F}_{p^2}\} = \{[x:y]: \ x,y \in \mathbb{F}_p, xy \neq 0\},$$

where $a+bi \mapsto [a+1,b]$, $[x:y] \mapsto \frac{x^2-y^2}{x^2+y^2} + \frac{2xy}{x^2+y^2}i$

Proj. Squaring (**2m**):	$[x:y]^2 = [(x+y)(x-y) : 2xy]$
Proj. Cubing (**2m+2s**):	$[x:y]^3 = [x(x^2-3y^2) : y(3x^2-y^2)]$
Proj. Mul. (**3m**):	$[x:y][z:t] = [xz-yt : (x+y)(z+t)-xz-yt]$
Mixed Mul. (**2m**):	$[x:y][\alpha,1] = [x\alpha - y : x + y\alpha]$
Inversion (**0m**):	$[x:y]^{-1} = [-x:y]$
Proj. Equality check (**2m**):	$[x:y] = [z:t] \iff xt - yz = 0$
Mixed Equality check (**1m**):	$[x:y] = [\alpha:1] \iff x - y\alpha = 0$

Remark

Traditional costs: squaring (**2s**), and cubing (**2m+1s**), group multiplication (**3m**), and free equality check

Table compression via torus representation

- Use $C(a+bi) = (a+1)/b$ to further compress T^{sgn} by a factor of 2:

$$CT[k][d] = C(T^{\text{sgn}}[k][d]) = C(g^{-d\ell^k})$$

- This yields an overall compression of tables by a factor of 4
- Computational overheads:
  - Right traversals $\Delta_{j,k+1} \leftarrow \Delta_{j,k} \cdot CT[j+k][d_k]$ become *mixed multiplications*

    $\ell = 2,3$: The cost **3m** changes to **2m**

  - Left traversals $\Delta_{j+1,k} \leftarrow \Delta_{j,k}^{\ell}$ become *projective exponentiations*

    $\ell = 2$: The cost **2s** changes to **2m**

    $\ell = 3$: The cost **2m+1s** changes to **2m+2s**

  - Table look ups to determine d_k via

    $$\Delta_{e-1-k,k} == T[e-1][d]^{-1} \Rightarrow d_k \leftarrow d$$

    used to be free but now requires $(\ell-1)/2$ multiplication at each leaf, and there are $(e-1)$ leaves

Computational overheads and our proposal

- Computational overheads for determining d_k using torus-based representations become non-trivial if PH with width-ω windows is used:

$$d = \sum_{i=0}^{e-1} d_i \ell^i, \; d_i \in [0, \ell)$$
$$= \sum_{i=0}^{m-1} D_i L^i, \; m = \lceil e/\omega \rceil, \; L = \ell^\omega, \; D_i \in [0, L)$$

$$\Delta_{m-1-k,k} == T[m-1][D]^{-1} \; \Rightarrow \; D_k \leftarrow D$$

- Now, it requires $(L-1)/2$ multiplications per leaf

Our proposal

Instead of going through $(L-1)/2$ equality checks, solve for the logarithm D_k of $\Delta_{m-1-k,k}$ base $\rho = g^{L^{m-1}}$, which generates $\mathbb{G}_{\ell,\omega}$

$$\Delta_{m-1-k,k} = T[m-1][D_k]^{-1} = (g^{L^{m-1}})^{D_k}$$

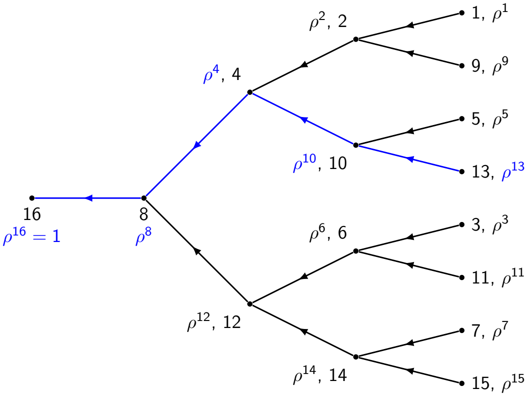
Discrete logarithms in $\mathbb{G}_{\ell,\omega}$

- Let $\mathbb{G}_{\ell,\omega} = \langle \rho \rangle$ be a cyclic group of order $L = \ell^\omega$, with torus-based representations
- Solving DLP with table look ups requires approximately $\ell^\omega/2$ $\mathbb{F}_p$-elements to store and $\ell^\omega/2$ equality checks, equivalently $\ell^\omega/2$ $\mathbb{F}_p$-multiplications (m)
- We propose new algorithms to solve DLP, with (mostly) linear complexity in ω

	Restriction	Average Complexity	Storage
Alg. 1	$\ell = 2$	$\left(\frac{7}{2}\omega - 4 + \frac{4}{2^\omega}\right)$ m	$2^\omega + \omega - 2$
Alg. 2	$\ell = 2$	$\left(3\omega - 4 + \frac{1}{2^{\omega-2}}\right)$ m	$2^{\omega-2} + \omega + 2$
Alg. 4	$\ell = 2$	$\left(2^{\omega-3} - \frac{1}{2}\right)$ m	$2^{\omega-1} + 1$
Alg. 1	$\ell = 3$	$\left(\frac{28}{5}\omega - \frac{33}{10} + \frac{33}{10\cdot 3^\omega}\right)$ m	$3^\omega + \omega - 2$
Alg. 3	$\ell = 3$	$\left(\frac{79}{15}\omega - \frac{33}{10} + \frac{33}{10\cdot 3^\omega}\right)$ m	$3^{\omega-1} + \omega - 1$

New algorithms to solve DLP in $\mathbb{G}_{\ell,\omega}$

- **Main ideas:**
  1. Construct an ℓ-ary like graph $\mathcal{G}_{\ell,\omega}$ of depth-ω such that an ℓ-exponentiation in $\mathbb{G}_{\ell,\omega}$ corresponds to traversing an edge in $\mathbb{G}_{\ell,\omega}$
  2. Given $h \in \mathbb{G}_{\ell,\omega}$, ℓ-exponentiate h until the result is identity. Extract the unique path in $\mathcal{G}_{\ell,\omega}$, so that the starting vertex yields the logarithm of h base ρ
- Example for $\ell = 2$, $\omega = 4$: $\mathbb{G}_{\ell,\omega} = \langle \rho \rangle$, $h = \rho^{13}$



Theoretical results

Theorem

Let $h \in \mathbb{G}_{\ell,\omega}$ be an element of order ℓ^k for some arbitrary $k \in \{1, \ldots, \omega\}$. Define a sequence $H = [h_0, \ldots, h_k]$ such that $h_k = h$ and $h_{j-1} = h_j^{\ell}$ for $j = 1, \ldots, k$. Then, there exists a unique path $P_{0,k} = v_{0,0}, v_{1,i_1}, \ldots, v_{k,i_k}$ in $\mathcal{G}_{\ell,\omega}$ such that $v_{j,i_j} \in V_j$ and $h_j = g_{j,i_j}$ for $j = 0, \ldots, k$.

Theorem

Let $1 \neq h \in \mathbb{G}_{\ell,\omega} = \langle \rho \rangle$. Given h and ρ, one can determine

1. *k, $i_1, i_2, \ldots, i_k$, that corresponds to the path $P_{0,k} = v_{0,0}, v_{1,i_1}, \ldots, v_{j,i_k}$ as in the above theorem;*
2. *$s_1, s_2, \ldots, s_k$ such that , $s_1 = i_1 + 1$, $s_j \in \{0, \ldots, (\ell - 1)\}$, and $i_j = \ell \cdot i_{j-1} + s_k$ for $j = 2, \ldots, k$.*

Moreover, $h = \rho^d$, where $d = \ell^{\omega-k} \sum_{j=1}^{k} s_j \ell^{j-1}$.

Refinements and other algorithms

1. Half of the paths in $\mathcal{G}_{\ell,\omega}$ can be eliminated at a cost of inverting elements in $\mathbb{G}_{\ell,\omega}$
2. We propose another algorithm for $\ell = 2$: exploit algebraic relations induced by the order of points in $\mathbb{G}_{\ell,\omega}$ and recursively enumerate them
3. Run time is exponential in ω but performs better than the previous one for small ω
4. Best of both worlds: a hybrid of the two algorithms

Parameters	Average Complexity
if $\omega_2 \leq 2$ and $\omega_2 \leq \omega_1$	$3\omega_2 + 2^{\omega_1 - 3} - \frac{1}{2} - \frac{\omega_2+2}{2^{\omega_1}} + \frac{2}{2^{\omega}}$
if $2 < \omega_2$ and $\omega_2 \leq \omega_1$	$3\omega_2 + 2^{\omega_1 - 3} - \frac{1}{2} + \frac{2^{\omega_2 - 3} - \omega_2 - \frac{5}{2}}{2^{\omega_1}} + \frac{2}{2^{\omega}}$
if $\omega_1 = 2$ and $2 < \omega_2$	$3\omega_2 - \frac{5}{4} + \frac{6}{2^{\omega}}$
if $2 < \omega_1$ and $\omega_1 < \omega_2$	$3\omega_2 + 2^{\omega_1 - 3} - 2^{\omega_1 - \omega_2 - 4} - \frac{5}{16} - \frac{\omega_1 + \frac{7}{2}}{2^{\omega_1}} + \frac{1}{2^{\omega_2}} + \frac{2}{2^{\omega}}$

Implementation results and comparisons

- $\ell = 2$ case enjoys the full factor-4 compression and slight speed-ups
- $\ell = 3$ case enjoys factor-2 compression but factor-4 compression suffers from a computational overhead: up to 9% in key generation
- A reasonable choice:
  - For $\ell = 2$, use signed-digits and torus representations (factor-4 compression)
  - For $\ell = 3$, use only signed-digits (factor-2 compression)
- New table sizes:
  - $\ell = 2$: 18KiB to 240KiB for all SIKE parameter sets
  - $\ell = 3$: 34KiB to 477KiB for all SIKE parameter sets

Implementation results and comparisons

- Comparative results of the average cost (in $\mathbb{F}_p$-multiplications) and table sizes (in KiB) to compute logarithms in $\mathbb{G}_{2,e}$ and $\mathbb{G}_{3,e}$
- Our results have been implemented in C and available at `https://github.com/microsoft/PQCrypto-SIDH`

DLP in $\mathbb{G}_{2,e}$	source	$\omega = 3$		$\omega = 4$		$\omega = 5$	
		time	size	time	size	time	size
$e = 216$ (SIKEp434)	previous	1944	70	1600	105	1415	342
	ours	1818	18	1542	27	1408	86
$e = 372$ (SIKEp751)	previous	3765	197	3126	296	2748	954
	ours	3476	49	2964	74	2688	240

DLP in $\mathbb{G}_{3,e}$	source	$\omega = 2$		$\omega = 3$		$\omega = 4$	
		time	size	time	size	time	size
$e = 137$ (SIKEp434)	previous	1845	151	1407	301	1185	688
	ours	2371	34	2029	73	1868	172
$e = 239$ (SIKEp751)	previous	3621	429	2793	859	2337	1932
	ours	4542	95	3886	207	3507	477

Thank You